

Replica Analysis for the Duality of the Portfolio Optimization Problem

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In the present paper, the primal-dual problem consisting of the investment risk minimization problem and the expected return maximization problem in the mean-variance model is discussed using replica analysis. As a natural extension of the investment risk minimization problem under only a budget constraint that we analyzed in a previous study, we herein consider a primal-dual problem in which the investment risk minimization problem with budget and expected return constraints is regarded as the primal problem, and the expected return maximization problem with budget and investment risk constraints is regarded as the dual problem. With respect to these optimal problems, we analyze a quenched disordered system involving both of these optimization problems using the approach developed in statistical mechanical informatics, and confirm that both optimal portfolios can possess the primal-dual structure. Finally, the results of numerical simulations are shown to validate the effectiveness of the proposed method.

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I. INTRODUCTION

The portfolio optimization problem, a mathematical finance problem, originating from the mean-variance model proposed by Markowitz in 1952 and 1959 [1, 2], and the optimal strategy of diversification investment of the expected investment risk and the expectation of investment return, has been solved in several operations research studies [3, 4]. However, these previous studies considered only the analysis of annealed disordered systems in the framework of many-body complex systems. The analysis of a quenched disordered system expected by the investor in practice has been investigated only rarely in operations research. Recently, analysis of quenched disordered systems involving the portfolio optimization problem has been performed using replica analysis, the belief propagation method, and the asymptotic eigenvalue distribution of a random matrix, which were developed in several previous studies in interdisciplinary research fields such as statistical mechanical informatics and econophysics [5–13]. For example, Ciliberti et al. performed a replica analysis of the minimal investment risk for the absolute deviation model and the expected shortfall model for a quenched disordered system, and reported the phase diagram for the expected shortfall model in the zero temperature limit [5, 6]. Kondor et al. evaluated the noise sensibility of the estimated error of the optimal solution with respect to several risk functions, such as variance, absolute deviation, expected shortfall, and maximum loss using numerical simulations [7]. Moreover, using numerical simulations, Pafka et al. showed that the optimal portfolio of each asset, which is the portfolio that can minimize the investment risk in the mean-variance model under only a budget constraint, has a normal distribution, and evaluated the ratio between the minimal expected investment risk and

the in-sample risk [8]. Shinzato reported that the minimal investment risk and its investment concentration in the mean-variance model under only a budget constraint are satisfied by the self-averaging property using a large-deviation approach and replica analysis, and performed a quenched disordered analysis of this investment system [9]. Furthermore, Shinzato et al. developed a fast algorithm for resolving the optimal solution that can minimize each investment risk, including that in the mean-variance model, the absolute deviation model, and the expected shortfall model, using the belief propagation method [10]. Varga-Haszonits et al. determined the optimal portfolio that can minimize the variance between the whole return in each scenario and the expected return under budget and expected return constraints in a quenched disordered system using replica analysis [11]. In addition, Shinzato performed an analysis of a quenched disordered system involving the investment risk minimization problem with budget and investment concentration constraints, and the investment concentration maximization and/or minimization problem with budget and investment risk using replica analysis, and clarified that the primal-dual structure also holds for a quenched system [12, 13].

Quenched analysis has been examined extensively. In particular, the portfolio optimization problem under several constraints has been analyzed in a number of studies [11–13]. However, the investment risk minimization problem with budget and expected return constraints, and the expected return maximization problem with budget and investment risk constraints, have rarely been investigated. Although Varga-Haszonits et al. solved the portfolio optimization problem under budget and expected return constraints using replica analysis [11], the object function considered was defined as half of the sum of the squares of the differences between the whole return in each scenario and the expected return, rather than using the variance of the whole return in each scenario. In other words, since the object function considered was not the investment risk, the investment risk

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minimization problem with budget and expected return constraints, which is a natural extension of our previous study [9], has not yet been examined. Furthermore, Shinzato first performed a replica analysis of the primal-dual problem with respect to the mean-variance model. From a unified viewpoint, it is also necessary to multidirectionally examine both portfolio optimization problems, i.e., the investment risk minimization problem with budget and expected return constraints and the expected return maximization problem with budget and investment risk constraints.

The goal of the present paper is to perform a replica analysis of a primal-dual problem consisting of the investment risk minimization problem and the expected return maximization problem in the mean-variance model, following the analytical approach used in previous studies [12, 13]. As a natural extension of our previous study [9], which only analyzed an investment risk minimization problem under a budget constraint, we herein consider a primal-dual problem in which the investment risk minimization problem with budget and expected return constraints is regarded as the primal problem and the expected return maximization problem with budget and investment risk constraints is regarded as the dual problem. With respect to these portfolio optimization problems, using an analytical approach in statistical mechanical informatics, we analyze a quenched disordered system and determine whether both optimal solutions can possess the primal-dual structure. Moreover, we compare with the results with those obtained through numerical simulations in order to validate the effectiveness of the proposed method.

The remainder of the present paper is organized as follows. In the next section, we describe the primal problem and the dual problem handled herein, where, for the sake of convenience, the primal problem is the investment risk minimization problem with budget and expected return constraints and the dual problem is the expected return maximization problem with budget and investment risk constraints. In Section III, following the analytical procedure used in our previous study [9], a replica analysis of a quenched disordered system involving the primal and dual problems is performed. Moreover, an annealed disordered system is analyzed using Lagrange's method of undetermined multipliers. In Section IV, in order to confirm the effectiveness of the proposed method, we compare the results with those of numerical experiments. Finally, Section V presents conclusions and areas for future research.

II. MODEL SETTING

First, we formulate the primal and dual problems in the mean-variance model considered herein.

A. Primal problem

Similar to previous studies [5–12], the present paper considers a stable investment market that can handle the investment of N assets. Here, w_i is a portfolio of assets $i (= 1, 2, \dots, N)$, which can be described as $\vec{w} = (w_1, \dots, w_N)^T \in \mathbf{R}^N$, where T represents the transpose of the matrix and/or vector. Moreover, in order to simplify the discussion, we assume that short selling, i.e., $w_i \leq 0$ is allowed. Next, given p scenarios (or periods), which are used in investing, the return rate of asset i in scenario $\mu (= 1, 2, \dots, p)$ can be described as $\bar{x}_{i\mu}$, and the return of each asset is assumed to be independently distributed. Based on this assumption, we do not consider period correlation. Furthermore, the mean and variance of the return rate of asset i are denoted as r_i and s^2 , respectively. Moreover, the mean r_i is assumed to be a hyperparameter that is independently and identically distributed with a probability distribution with a mean and variance of m and σ^2 , respectively. Thus, the investment risk $\mathcal{H}(\vec{w}|X)$ with respect to portfolio \vec{w} is defined as follows:

$$\begin{aligned} \mathcal{H}(\vec{w}|X) &= \frac{1}{2N} \sum_{\mu=1}^p \left(\sum_{i=1}^N \bar{x}_{i\mu} w_i - \sum_{i=1}^N r_i w_i \right)^2 \\ &= \frac{1}{2} \sum_{\mu=1}^p \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N x_{i\mu} w_i \right)^2, \end{aligned} \quad (1)$$

where the modified return rate $x_{i\mu} = \bar{x}_{i\mu} - r_i$ is used. Then, the mean and variance of the modified return rate are 0 and s^2 , respectively. Furthermore, the return rate matrix is given by $X = \left\{ \frac{x_{i\mu}}{\sqrt{N}} \right\} \in \mathbf{R}^{N \times p}$. With respect to this object function, as budget and expected return constraints,

$$N = \sum_{i=1}^N w_i, \quad (2)$$

$$NR = \sum_{i=1}^N r_i w_i, \quad (3)$$

are used. The budget constraint in Eq. (2) was used in previous studies [5, 6, 9–12], and Eq. (3) describes the expected return constraints. Moreover, R is a coefficient characterizing the expected return. In addition, the feasible portfolio subset $\mathcal{P}(R)$ is given in terms of portfolio \vec{w} and satisfies these two constraints:

$$\mathcal{P}(R) = \{ \vec{w} \in \mathbf{R}^N \mid N = \vec{e}^T \vec{w}, NR = \vec{r}^T \vec{w} \}, \quad (4)$$

where $\vec{e} = (1, \dots, 1)^T \in \mathbf{R}^N$ is the constant vector and $\vec{r} = (r_1, r_2, \dots, r_N)^T \in \mathbf{R}^N$ is the mean vector. Thus, the primal problem considered herein is the problem of determining the optimal portfolio \vec{w} that can minimize the investment risk $\mathcal{H}(\vec{w}|X)$ in Eq. (1) in the feasible

portfolio subset $\mathcal{P}(R)$ in Eq. (4). The minimal investment risk per asset ε is calculated as follows:

$$\varepsilon = \min_{\vec{w} \in \mathcal{P}(R)} \left\{ \frac{1}{N} \mathcal{H}(\vec{w}|X) \right\}. \quad (5)$$

Based on the argument in our previous study [9], in the investment risk minimization problem imposing only a budget constraint, the minimal investment risk per asset is $\varepsilon = \frac{s^2(\alpha-1)}{2}$, where the scenario ratio $\alpha = p/N$ is used.

When the rank of the return rate matrix $X = \left\{ \frac{x_{i\mu}}{\sqrt{N}} \right\} \in \mathbf{R}^{N \times p}$, $\text{rank}(X) = N$, i.e., $p > N$, because the optimal solutions of the primal problem and the dual problem described later herein can be determined uniquely, and the argument is limited to $\alpha = p/N > 1$.

Finally, there are two important considerations. First, using Lagrange's method of undetermined multipliers,

$$\varepsilon = \frac{N}{2\vec{e}^T J^{-1} \vec{e}} \left\{ 1 + \frac{\left(R - \frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2}{\frac{\vec{r}^T J^{-1} \vec{r}}{\vec{e}^T J^{-1} \vec{e}} - \left(\frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2} \right\}, \quad (6)$$

is solved analytically, where the Wishart matrix $J = XX^T \in \mathbf{R}^{N \times N}$ is used. (See Appendix A for details.) However, the computational complexity of the inverse of the Wishart matrix $J = XX^T$ must be $O(N^3)$, and, in the analysis of a quenched disordered system, it is necessary to average Eq. (6) over the return rate matrix X and the mean vector \vec{r} . Therefore, it is not easy to directly evaluate the configuration average $E_{X,\vec{r}}[\varepsilon]$, where $E_{X,\vec{r}}[f(X,\vec{r})]$ describes the expectation of $f(X,\vec{r})$ over the whole configuration of X and \vec{r} . As such, we will analyze the configuration average of the minimal investment risk per asset using replica analysis, which does not directly require evaluation of the inverse matrix in order to solve the configuration average.

Next, based on the definitions of the investment risk and expected return, we can separate the randomness in the expected return from the randomness in the investment risk. In other words, in the mean-variance model, since the investment risk is defined as half of the sum of the squares of the differences between whole return $\sum_{i=1}^N \bar{x}_{i\mu} w_i$ in scenario μ and expected return $\sum_{i=1}^N r_i w_i$, using the modified return rate $x_{i\mu}$, we can remove the influence of the randomness of the expected return from the investment risk. Moreover, based on this formulation, the distributions of $x_{i\mu}$ and r_i can be set independently.

B. Dual problem

In Subsection II A, similar to the primal problem, we formulate the dual problem as obtaining the optimal portfolio \vec{w} that can maximize the expected return with imposed budget and investment risk constraints. The expected return $\mathcal{H}'(\vec{w}|\vec{r})$ is defined as follows:

$$\mathcal{H}'(\vec{w}|\vec{r}) = \sum_{i=1}^N r_i w_i, \quad (7)$$

where

$$N = \sum_{i=1}^N w_i, \quad (8)$$

$$N\varepsilon' = \frac{1}{2} \sum_{\mu=1}^p \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N w_i x_{i\mu} \right)^2, \quad (9)$$

are the budget and investment risk constraints. Equation (8) is identical to the budget constraint given in Eq. (2), and Eq. (9) describes the investment risk constraint. Moreover, ε' is a coefficient characterizing the investment risk, and the feasible portfolio subset $\mathcal{D}(\varepsilon')$ is defined in terms of the portfolio \vec{w} , which satisfies these two constraints:

$$\mathcal{D}(\varepsilon') = \left\{ \vec{w} \in \mathbf{R}^N \left| N = \vec{e}^T \vec{w}, N\varepsilon' = \frac{1}{2} \vec{w}^T J \vec{w} \right. \right\}. \quad (10)$$

Thus, the expected return per asset of the dual problem, R' , is calculated as follows:

$$R' = \max_{\vec{w} \in \mathcal{D}(\varepsilon')} \left\{ \frac{1}{N} \mathcal{H}'(\vec{w}|\vec{r}) \right\}. \quad (11)$$

Moreover, using Lagrange's method of undetermined multipliers with respect to this dual problem,

$$R' = \sqrt{\frac{\vec{r}^T J^{-1} \vec{r}}{\vec{e}^T J^{-1} \vec{e}} - \left(\frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2} \sqrt{\frac{2\varepsilon' \vec{e}^T J^{-1} \vec{e}}{N} - 1} + \frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}}, \quad (12)$$

is solved analytically. (See Appendix B for details.) However, since it is also difficult to directly assess Eq. (12) for a quenched disordered system, replica analysis is used.

Finally, there are two important considerations. First, the object function in Eq. (1) in the primal problem corresponds to the second constraint in Eq. (9) in the dual problem, and the object function in Eq. (7) in the dual problem corresponds to the second constraint in Eq. (3) in the primal problem. Second, the primal problem considered in the present paper is a natural extension of that in our previous study [9]. However, since we consider the expected return constraint in the present study, the proposed method is more practicable. The portfolio optimization problem with budget and expected return constraints has also been discussed [11]. However, since in that study, the object function was not the investment risk, and the randomness of the asset returns in the investment risk and the expected return were different, it was not always a natural extension of our previous study [9]. Furthermore, since the primal-dual structure is not handled analytically, it is more important to support the results of Varga-Haszonits et al. [11] by our findings in the present paper.

III. REPLICANALYSIS

A. Replica analysis for the primal problem

In this section, we perform a replica analysis of a quenched disordered system involving the primal problem. Following our previous study [9], the partition function for the canonical ensemble of this investment system of inverse temperature β , $Z(R, X, \vec{r})$ is denoted as follows:

$$Z(R, X, \vec{r}) = \int_{\vec{w} \in \mathcal{P}(R)} d\vec{w} e^{-\beta \mathcal{H}(\vec{w}|X)}, \quad (13)$$

where the investment risk $\mathcal{H}(\vec{w}|X)$ in Eq. (1) is regarded as the Hamiltonian, and the integral of \vec{w} is regarded as the feasible portfolio subset $\mathcal{P}(R)$.

Thus, the minimal investment risk per asset ε is calculated from the following thermodynamic relation:

$$\begin{aligned} \varepsilon &= \min_{\vec{w} \in \mathcal{P}(R)} \left\{ \frac{1}{N} \mathcal{H}(\vec{w}|X) \right\} \\ &= - \lim_{\beta \rightarrow \infty} \frac{1}{N} \frac{\partial}{\partial \beta} \log Z(R, X, \vec{r}). \end{aligned} \quad (14)$$

The analysis of a quenched disordered system is performed as follows:

$$\begin{aligned} \phi(R) &= \lim_{N \rightarrow \infty} \frac{1}{N} E_{X, \vec{r}} [\log Z(R, X, \vec{r})] \\ &= \text{Extr}_{\Theta} \left\{ \frac{1}{2} (\chi_w + q_w) (\tilde{\chi}_w - \tilde{q}_w) + \frac{q_w \tilde{q}_w}{2} \right. \\ &\quad \left. - k - (R - m)\theta + \frac{\sigma^2 \theta^2}{2} \chi_w - \frac{\alpha}{2} \log(1 + \beta s^2 \chi_w) \right. \\ &\quad \left. - \frac{\alpha \beta s^2 q_w}{2(1 + \beta s^2 \chi_w)} - \frac{1}{2} \log \tilde{\chi}_w + \frac{\tilde{q}_w + k^2}{2\tilde{\chi}_w} \right\}, \end{aligned} \quad (15)$$

where $\Theta = \{k, \theta, \chi_w, q_w, \tilde{\chi}_w, \tilde{q}_w\}$ is used. Moreover, the notation $\text{Extr}_z f(z)$ describes the extremum of function $f(z)$ with respect to the variable z . (See Appendix C for details.)

From these extremum conditions, as the results of the principal variables at inverse temperature β , we obtain

$$\varepsilon = \frac{1}{2\beta} + \frac{s^2(\alpha - 1)}{2} \left(1 + \frac{(R - m)^2}{\sigma^2} \right), \quad (16)$$

$$q_w = \frac{\alpha}{\alpha - 1} \left(1 + \frac{(R - m)^2}{\sigma^2} \right), \quad (17)$$

$$\chi_w = \frac{1}{\beta s^2(\alpha - 1)}. \quad (18)$$

In the zero temperature limit, we obtain

$$\varepsilon = \frac{s^2(\alpha - 1)}{2} \left(1 + \frac{(R - m)^2}{\sigma^2} \right), \quad (19)$$

where the investment risk per asset ε is a quadratic function of R . In addition, when $R = m$, the minimum value of ε is $\frac{s^2(\alpha - 1)}{2}$. Now, if $R = m$ in Eq. (3), since the

expected return constraint is consistent with the budget constraint in Eq. (2), in practice, this result is consistent with the minimization of the investment risk with only a budget constraint being imposed.

Moreover, the Sharpe ratio, which measures the expected return with respect to the investment risk, i.e., $S = \frac{R}{\sqrt{2\varepsilon}}$, is derived as follows:

$$S = \frac{1}{s\sqrt{\alpha - 1}} \frac{R}{\sqrt{1 + \frac{(R - m)^2}{\sigma^2}}}. \quad (20)$$

From $\frac{\partial S}{\partial R} = 0$, $R = m + \frac{\sigma^2}{m}$ and $\varepsilon = \frac{s^2(\alpha - 1)}{2} \left(1 + \frac{\sigma^2}{m^2} \right)$ are calculated, and the maximum Sharpe ratio S_{\max} is then obtained as follows:

$$S_{\max} = \frac{\sqrt{m^2 + \sigma^2}}{s\sqrt{\alpha - 1}}. \quad (21)$$

Finally, we can also discuss the analysis of the annealed disordered system. We have

$$\varepsilon^{\text{OR}} = \frac{s^2\alpha}{2} \left(1 + \frac{(R - m)^2}{\sigma^2} \right), \quad (22)$$

$$q_w^{\text{OR}} = 1 + \frac{(R - m)^2}{\sigma^2}. \quad (23)$$

(See Appendix E for details.) As the relationship between the minimal investment risk per asset ε in Eq. (19) and the minimal expected investment risk per asset ε^{OR} in Eq. (22),

$$\varepsilon < \varepsilon^{\text{OR}}, \quad (24)$$

is obtained. Similarly, $S^{\text{OR}} = \frac{R}{\sqrt{2\varepsilon^{\text{OR}}}} < S$ also holds.

B. Replica analysis for the dual problem

In this subsection, we describe a replica analysis of a quenched disordered system involving a dual problem. Following the above approach, the partition function for the canonical ensemble of this investment system with an inverse temperature β , $Z(\varepsilon', X, \vec{r})$, is defined as follows:

$$Z(\varepsilon', X, \vec{r}) = \int_{\vec{w} \in \mathcal{D}(\varepsilon')} d\vec{w} e^{\beta \mathcal{H}'(\vec{w}|\vec{r})}, \quad (25)$$

where the expected return $\mathcal{H}'(\vec{w}|\vec{r})$ in Eq. (7) is regarded as the Hamiltonian, and the integral of \vec{w} is regarded as the feasible portfolio subset $\mathcal{D}(\varepsilon')$.

Then, the maximum expected return per asset R' is derived from the following thermodynamic relation:

$$\begin{aligned} R' &= \max_{\vec{w} \in \mathcal{D}(\varepsilon')} \left\{ \frac{1}{N} \mathcal{H}'(\vec{w}|\vec{r}) \right\} \\ &= \lim_{\beta \rightarrow \infty} \frac{1}{N} \frac{\partial}{\partial \beta} \log Z(\varepsilon', X, \vec{r}), \end{aligned} \quad (26)$$

where in order to maximize the expected return $\mathcal{H}'(\vec{w}|\vec{r})$ in the dual problem, we do not use the description of

the Boltzmann factor given in Eq. (13), but rather use that presented in Eq. (25). Note that we also use the thermodynamic relation given in Eq. (26).

Then, the analysis of the quenched disordered system is performed as follows:

$$\begin{aligned}\phi(\varepsilon') &= \lim_{N \rightarrow \infty} \frac{1}{N} E_{X, \vec{r}} [\log Z(\varepsilon', X, \vec{r})] \\ &= \text{Extr}_{\Theta} \left\{ \frac{1}{2} (\chi_w + q_w) (\tilde{\chi}_w - \tilde{q}_w) + \frac{q_w \tilde{q}_w}{2} \right. \\ &\quad - k + \theta \varepsilon' + \beta m + \frac{\sigma^2 \beta^2}{2} \chi_w - \frac{\alpha}{2} \log(1 + \theta s^2 \chi_w) \\ &\quad \left. - \frac{\alpha \theta s^2 q_w}{2(1 + \theta s^2 \chi_w)} - \frac{1}{2} \log \tilde{\chi}_w + \frac{\tilde{q}_w + k^2}{2 \tilde{\chi}_w} \right\}. \quad (27)\end{aligned}$$

(See Appendix D in for details.)

From the extremum conditions, as the results of the principal variables at inverse temperature β , we have

$$R' = m + \sigma(\beta \sigma \chi_w), \quad (28)$$

$$q_w = \frac{\alpha}{\alpha - 1} (1 + \beta^2 \sigma^2 \chi_w^2), \quad (29)$$

$$\chi_w = \frac{1}{\theta s^2 (\alpha - 1)}. \quad (30)$$

Furthermore, from $\varepsilon' = \frac{\alpha s^2 \chi_w}{2(1 + \theta s^2 \chi_w)} + \frac{\alpha s^2 q_w}{2(1 + \theta s^2 \chi_w)^2} = \frac{1}{2\theta} + \frac{s^2(\alpha-1)}{2} (1 + \beta^2 \sigma^2 \chi_w^2)$, β and θ are satisfied by the following relation:

$$\left(\frac{\beta \sigma}{\theta s^2 (\alpha - 1)} \right)^2 = \frac{2}{s^2 (\alpha - 1)} \left(\varepsilon' - \frac{1}{2\theta} \right) - 1. \quad (31)$$

In the zero temperature limit, since the right-hand side is $O(1)$, $\beta/\theta \sim O(1)$ holds. Then,

$$R' = m + \sigma \sqrt{\frac{2\varepsilon'}{s^2(\alpha-1)}} - 1, \quad (32)$$

$$q_w = \frac{\alpha}{\alpha - 1} \frac{2\varepsilon'}{s^2(\alpha-1)}, \quad (33)$$

are obtained. Moreover, the Sharpe ratio $S = \frac{R'}{\sqrt{2\varepsilon'}}$ is solved as follows:

$$S = \frac{m + \sigma \sqrt{\frac{2\varepsilon'}{s^2(\alpha-1)}} - 1}{\sqrt{2\varepsilon'}}. \quad (34)$$

In addition, from $\frac{\partial S}{\partial \varepsilon'} = 0$, $\varepsilon' = \frac{s^2(\alpha-1)}{2} \left(1 + \frac{\sigma^2}{m^2} \right)$ and $R' = m + \frac{\sigma^2}{m}$ are calculated. Then, the maximum Sharpe ratio S_{\max} is given as

$$S_{\max} = \frac{\sqrt{m^2 + \sigma^2}}{s\sqrt{\alpha-1}}. \quad (35)$$

Finally, three points should be noted here. First, the previous subsection and this subsection describe the primal-dual structure. When we derive R from Eq. (19), and set $R = R'$ and $\varepsilon = \varepsilon'$, Eq. (32) is obtained. Similarly, $R = R'$ and $\varepsilon = \varepsilon'$ are set, and q_w in Eq. (17)

is consistent with q_w in Eq. (33). In other words, for a quenched disordered system, the optimal portfolio that can minimize the investment risk under a fixed expected return is consistent with the optimal portfolio that can maximize the expected return under a fixed investment risk.

Next, the optimal portfolio that can minimize the expected return under a fixed investment risk is also solved using replica analysis. Therefore, the minimum expected return per asset R'' is

$$\begin{aligned}R'' &= \min_{\vec{w} \in \mathcal{D}(\varepsilon')} \left\{ \frac{1}{N} \mathcal{H}'(\vec{w} | \vec{r}) \right\} \\ &= \lim_{\beta \rightarrow -\infty} \frac{1}{N} \frac{\partial}{\partial \beta} \log Z(\varepsilon', X, \vec{r}) \\ &= m - \sigma \sqrt{\frac{2\varepsilon'}{s^2(\alpha-1)}} - 1. \quad (36)\end{aligned}$$

In other words, there exists a portfolio for which the expected return is not less than Eq. (36).

Finally, we discuss the analysis of an annealed disordered system:

$$R'^{\text{OR}} = m + \sigma \sqrt{\frac{2\varepsilon'}{s^2\alpha}} - 1, \quad (37)$$

$$q_w^{\text{OR}} = \frac{2\varepsilon'}{s^2\alpha}. \quad (38)$$

(See Appendix F for details.) From Eqs. (37) and (38), we obtain

$$q_w^{\text{OR}} = 1 + \frac{(R' - m)^2}{\sigma^2}, \quad (39)$$

which is consistent with the finding in Eq. (23). In addition, from Eqs. (32) and (37), $R' > R'^{\text{OR}}$ holds, and Sharpe ratios $S = \frac{R'}{\sqrt{2\varepsilon'}}$ and $S^{\text{OR}} = \frac{R'^{\text{OR}}}{\sqrt{2\varepsilon'}}$ are confirmed to be satisfied for the case in which $S > S^{\text{OR}}$.

IV. NUMERICAL EXPERIMENTS

In this section, we investigate the validity of the proposed method through numerical experiments. The Wishart matrix $J = XX^T \in \mathbf{R}^{N \times N}$ can be defined in terms of the return rate matrix X and the mean vector \vec{r} , as follows:

$$\varepsilon(R, X, \vec{r}) = \frac{N}{2\vec{e}^T J^{-1} \vec{e}} \left\{ 1 + \frac{\left(R - \frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2}{\frac{\vec{r}^T J^{-1} \vec{r}}{\vec{e}^T J^{-1} \vec{e}} - \left(\frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2} \right\}, \quad (40)$$

$$\begin{aligned}R'(\varepsilon', X, \vec{r}) &= \sqrt{\frac{\vec{r}^T J^{-1} \vec{r}}{\vec{e}^T J^{-1} \vec{e}} - \left(\frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2} \sqrt{\frac{2\varepsilon' \vec{e}^T J^{-1} \vec{e}}{N}} - 1 \\ &\quad + \frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}}. \quad (41)\end{aligned}$$

Based on this, the C return rate matrices, $X^1, X^2, \dots, X^C \in \mathbf{R}^{N \times p}$ and the C mean vectors, $\bar{r}^1, \bar{r}^2, \dots, \bar{r}^C \in \mathbf{R}^N$,

$$\varepsilon = \frac{1}{C} \sum_{c=1}^C \varepsilon(R, X^c, \bar{r}^c), \quad (42)$$

$$R' = \frac{1}{C} \sum_{c=1}^C R'(\varepsilon', X^c, \bar{r}^c), \quad (43)$$

are estimated, where the elements of the c th return rate matrix $X^c = \left\{ \frac{x_{i\mu}^c}{\sqrt{N}} \right\} \in \mathbf{R}^{N \times p}$, $x_{i\mu}^c$, have independent and identical probability distributions having a mean of 0 and a variance of s^2 , and the component of the c th mean vector $\bar{r}^c = (r_1^c, \dots, r_N^c)^T \in \mathbf{R}^N$, r_i^c has an independent and identical probability distribution having a mean of m and a variance of σ^2 . Moreover, the investment concentration q_w and the Sharpe ratio S are also estimated.

Thus, in the numerical simulations, $N = 1,000$ and $p = 3,000$, i.e., $\alpha = p/N = 3$, the primal problem and the dual problem at $(s^2, m, \sigma^2) = (1, 1, 1)$ are examined. The sample size used in the estimation is $C = 100$. The results of the primal problem and the dual problem are shown in Figs. 1 and 2, respectively. In Fig. 1, the horizontal axis shows the return coefficient R , and the vertical axes show (a) the minimal investment risk per asset ε , (b) the investment concentration q_w , and (c) the Sharpe ratio S . Moreover, in Fig. 2, the horizontal axis shows the risk coefficient ε' , and the vertical axes show (a) the maximal expected return per asset R' , (b) the investment concentration q_w , and (c) the Sharpe ratio S . The solid (orange) lines indicate the results of the replica analysis, and the (blue) asterisks with the error bars show the numerical results. The dashed (black) lines in Fig. 1 indicate (a) $\frac{s^2(\alpha-1)}{2}$, (b) $\frac{\alpha}{\alpha-1}$, and (c) S_{\max} , and those in Fig. 2 indicate (a) m , (b) $\frac{\alpha}{\alpha-1}$, and (c) S_{\max} . As shown in these figures, the results derived by the proposed method are consistent with the numerical results, i.e., the effectiveness of the proposed approach is confirmed.

V. CONCLUSION AND FUTURE RESEARCH

In the present paper, in order to extend the portfolio optimization problem of a quenched disordered system with only a budget constraint, which has been considered in our previous study [9], we analyzed the portfolio optimization problem of a quenched disordered system with several constraints using replica analysis and discussed the primal-dual structure of the mean-variance model. In our previous studies [12, 13], the primal-dual structure with respect to investment concentration and investment risk was assessed. In the present paper, the portfolio optimization problem minimizing the investment risk with budget and expected return constraints is regarded as the primal problem, and the portfolio optimization problem

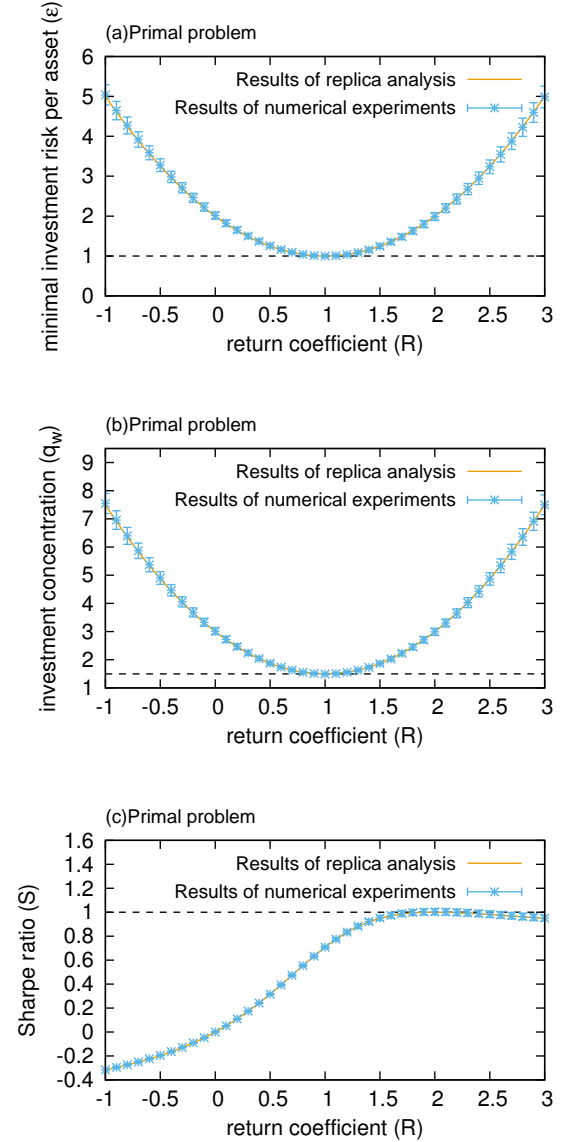


FIG. 1. Results of the replica analysis and the numerical experiments ($\alpha = p/N = 3$). The horizontal axis shows the return coefficient R , and the vertical axes show (a) the minimal investment risk per asset ε , (b) the investment concentration q_w , and (c) the Sharpe ratio S . The solid (orange) lines indicate the results of the replica analysis for (a) Eq. (19), (b) Eq. (17), and (c) Eq. (20). The (blue) asterisks with the error bars indicate the results of the numerical simulation, and the dashed (black) lines indicate the results for (a) $\frac{s^2(\alpha-1)}{2}$, (b) $\frac{\alpha}{\alpha-1}$, and (c) S_{\max} .

maximizing the expected return with budget and investment risk constraints is regarded as the dual problem. We clarified the primal-dual structure in these two portfolio optimization problems. Similar to the annealed disordered system considered in general operations research studies, the minimal investment risk was confirmed to

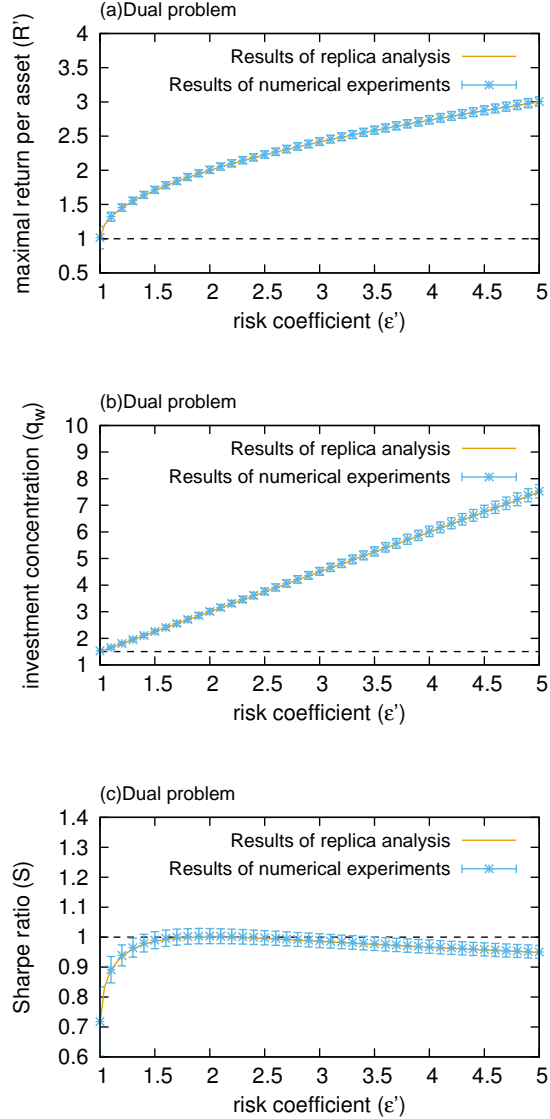


FIG. 2. Results obtained by the replica analysis and the numerical experiments ($\alpha = p/N = 3$). The horizontal axis shows the risk coefficient ϵ' , and the vertical axes show (a) the maximal return per asset R' , (b) the investment concentration q_w , and (c) the Sharpe ratio S . The solid (orange) lines show the results of the replica analysis for (a) Eq. (32), (b) Eq. (33), and (c) Eq. (34). The (blue) asterisks with the error bars indicate the results of the numerical simulation, and the dashed (black) lines indicate (a) m , (b) $\frac{\alpha}{\alpha-1}$, and (c) S_{\max} .

be a quadratic function with respect to the coefficient of the expected return constraint in the primal problem of a quenched disordered system. Moreover, in order to validate the effectiveness of the proposed method, we compared its results to those of numerical simulations and confirmed that there was good agreement.

In the future, since the randomness of the return rate

in the study by Varga-Haszonits et al. [11] is different from that in the present paper, we need to consider the primal-dual problem in terms of the randomness used in that study and theoretically develop a methodology for resolving the portfolio optimization problem. Moreover, in such cases, we also need to verify the mathematical structure of the Sharpe ratio.

Appendix A: Lagrange multiplier method for the primal problem

In this appendix, we discuss the portfolio optimization problem by applying Lagrange's method of undetermined multipliers to the primal problem. First, the Lagrange undetermined multiplier function is given as follows:

$$L = \frac{1}{2} \vec{w}^T J \vec{w} + k (N - \vec{w}^T \vec{e}) + \theta (NR - \vec{w}^T \vec{r}), \quad (\text{A1})$$

where the auxiliary variables k and θ are used. Since $\frac{\partial L}{\partial \vec{w}} = 0$, we obtain

$$\vec{w} = k J^{-1} \vec{e} + \theta J^{-1} \vec{r}, \quad (\text{A2})$$

and, since $\frac{\partial L}{\partial k} = \frac{\partial L}{\partial \theta} = 0$, we obtain

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \begin{pmatrix} \frac{\vec{e}^T J^{-1} \vec{e}}{\vec{r}^T J^{-1} \vec{e}} & \frac{\vec{r}^T J^{-1} \vec{e}}{\vec{r}^T J^{-1} \vec{r}} \\ \frac{\vec{r}^T J^{-1} \vec{e}}{N} & \frac{\vec{r}^T J^{-1} \vec{r}}{N} \end{pmatrix} \begin{pmatrix} k \\ \theta \end{pmatrix}, \quad (\text{A3})$$

where

$$\begin{pmatrix} k \\ \theta \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \frac{\vec{r}^T J^{-1} \vec{r}}{N} & -\frac{\vec{r}^T J^{-1} \vec{e}}{N} \\ -\frac{\vec{r}^T J^{-1} \vec{e}}{N} & \frac{\vec{e}^T J^{-1} \vec{e}}{N} \end{pmatrix} \begin{pmatrix} 1 \\ R \end{pmatrix}, \quad (\text{A4})$$

$$D = \left(\frac{\vec{e}^T J^{-1} \vec{e}}{N} \right)^2 \left\{ \frac{\vec{r}^T J^{-1} \vec{r}}{\vec{e}^T J^{-1} \vec{e}} - \left(\frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2 \right\}. \quad (\text{A5})$$

In addition, from Eq. (A2), we have $J \vec{w} = k \vec{e} + \theta \vec{r}$. Then, from $\epsilon = \frac{1}{2N} \vec{w}^T J \vec{w} = \frac{k + R\theta}{2}$,

$$\epsilon = \frac{N}{2 \vec{e}^T J^{-1} \vec{e}} \left\{ 1 + \frac{\left(R - \frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2}{\frac{\vec{r}^T J^{-1} \vec{r}}{\vec{e}^T J^{-1} \vec{e}} - \left(\frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2} \right\}, \quad (\text{A6})$$

is solved. This finding depends on the given return rate matrix X and mean vector \vec{r} . Next, we briefly consider the analysis of a quenched disordered system.

When N is sufficiently large, both sides of Eq. (A3) are averaged by the mean vector \vec{r} . Then, we obtain

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = E_{\vec{r}} \left[\begin{pmatrix} \frac{\vec{e}^T J^{-1} \vec{e}}{N} & \frac{\vec{r}^T J^{-1} \vec{e}}{N} \\ \frac{\vec{r}^T J^{-1} \vec{e}}{N} & \frac{\vec{r}^T J^{-1} \vec{r}}{N} \end{pmatrix} \begin{pmatrix} k \\ \theta \end{pmatrix} \right]. \quad (\text{A7})$$

Each component of the coefficient matrix can be obtained independently as

$$E_{\vec{r}} [\vec{r}^T J^{-1} \vec{e}] = m \vec{e}^T J^{-1} \vec{e}, \quad (\text{A8})$$

$$E_{\vec{r}} [\vec{r}^T J^{-1} \vec{r}] = m^2 \vec{e}^T J^{-1} \vec{e} + \sigma^2 \text{Tr} J^{-1} \quad (\text{A9})$$

in terms of the mean vector \vec{r} . Furthermore, using N eigenvalues of the Wishart matrix $J = XX^T \in \mathbf{R}^{N \times N}$, $\lambda_1, \dots, \lambda_N$, since $\sum_{i=1}^N \lambda_i^{-1} = \vec{e}^T J^{-1} \vec{e} = \text{Tr} J^{-1}$ holds, we have

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \frac{\vec{e}^T J^{-1} \vec{e}}{N} \begin{pmatrix} 1 & m \\ m & m^2 + \sigma^2 \end{pmatrix} \begin{pmatrix} k \\ \theta \end{pmatrix}. \quad (\text{A10})$$

In addition, from [9], we have

$$\frac{1}{N} \vec{e}^T J^{-1} \vec{e} = \frac{1}{s^2(\alpha - 1)}, \quad (\text{A11})$$

i.e., we obtain

$$\varepsilon = \frac{s^2(\alpha - 1)}{2} \left(1 + \frac{(R - m)^2}{\sigma^2} \right). \quad (\text{A12})$$

This finding is consistent with the results of the replica analysis in Eq. (19).

Appendix B: Lagrange multiplier method for the dual problem

Here, we analyze the portfolio optimization by applying Lagrange's method of undetermined multipliers to the dual problem. First, the Lagrange undetermined multiplier function is defined as follows:

$$L = \vec{r}^T \vec{w} + k(\vec{w}^T \vec{e} - N) + \theta \left(N\varepsilon' - \frac{1}{2} \vec{w}^T J \vec{w} \right), \quad (\text{B1})$$

where the auxiliary variables k and θ are used. Since $\frac{\partial L}{\partial \vec{w}} = 0$, we obtain

$$\vec{w} = \frac{k}{\theta} J^{-1} \vec{e} + \frac{1}{\theta} J^{-1} \vec{r}, \quad (\text{B2})$$

and since $\frac{\partial L}{\partial k} = \frac{\partial L}{\partial \theta} = 0$, we have

$$1 = \frac{k}{\theta} \frac{\vec{e}^T J^{-1} \vec{e}}{N} + \frac{1}{\theta} \frac{\vec{r}^T J^{-1} \vec{e}}{N}, \quad (\text{B3})$$

$$\begin{aligned} \varepsilon' &= \frac{1}{2} \left(\frac{k^2}{\theta^2} \frac{\vec{e}^T J^{-1} \vec{e}}{N} + 2 \frac{k}{\theta^2} \frac{\vec{r}^T J^{-1} \vec{e}}{N} + \frac{1}{\theta^2} \frac{\vec{r}^T J^{-1} \vec{r}}{N} \right), \\ &= \frac{N}{2 \vec{e}^T J^{-1} \vec{e}} \left[\left(\frac{k}{\theta} \frac{\vec{e}^T J^{-1} \vec{e}}{N} + \frac{1}{\theta} \frac{\vec{r}^T J^{-1} \vec{e}}{N} \right)^2 \right. \\ &\quad \left. + \frac{1}{\theta^2} \left(\frac{\vec{e}^T J^{-1} \vec{e}}{N} \right)^2 \left\{ \frac{\vec{r}^T J^{-1} \vec{r}}{\vec{e}^T J^{-1} \vec{e}} - \left(\frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2 \right\} \right]. \quad (\text{B4}) \end{aligned}$$

In summary,

$$\theta = \frac{\sqrt{D}}{\sqrt{2\varepsilon' \frac{\vec{e}^T J^{-1} \vec{e}}{N} - 1}}, \quad (\text{B5})$$

is obtained, where D is as defined in Eq. (A5). Moreover,

$$k = \frac{N}{\vec{e}^T J^{-1} \vec{e}} \left(\theta - \frac{\vec{r}^T J^{-1} \vec{e}}{N} \right) \quad (\text{B6})$$

is derived from Eq. (B3). In addition, using Eq. (B2), from $N\varepsilon' = \frac{1}{2} \vec{w}^T \left(\frac{k}{\theta} \vec{e} + \frac{1}{\theta} \vec{r} \right) = \frac{N}{2\theta} (k + R')$, we have

$$\begin{aligned} R' &= 2\varepsilon' \theta - k \\ &= \sqrt{\frac{\vec{r}^T J^{-1} \vec{r}}{\vec{e}^T J^{-1} \vec{e}} - \left(\frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}} \right)^2} \sqrt{\frac{2\varepsilon' \vec{e}^T J^{-1} \vec{e}}{N} - 1} \\ &\quad + \frac{\vec{r}^T J^{-1} \vec{e}}{\vec{e}^T J^{-1} \vec{e}}. \quad (\text{B7}) \end{aligned}$$

This finding depends on a given return rate matrix X and mean vector \vec{r} . Next, we briefly consider the analysis of a quenched disordered system.

If N is sufficiently large, both sides in Eqs. (B3) and (B4) are averaged over the mean vector. Then, we obtain

$$1 = \frac{\vec{e}^T J^{-1} \vec{e}}{N} \frac{k + m}{\theta}, \quad (\text{B8})$$

$$\varepsilon' = \frac{\vec{e}^T J^{-1} \vec{e}}{2N} \left(\frac{k^2}{\theta^2} + 2m \frac{k}{\theta^2} + \frac{1}{\theta^2} (m^2 + \sigma^2) \right). \quad (\text{B9})$$

In other words,

$$k = \frac{N\theta}{\vec{e}^T J^{-1} \vec{e}} - m, \quad (\text{B10})$$

$$\theta = \frac{\sigma \vec{e}^T J^{-1} \vec{e}}{N} \frac{1}{\sqrt{\frac{2\varepsilon' \vec{e}^T J^{-1} \vec{e}}{N} - 1}}, \quad (\text{B11})$$

are derived. Thus, $R' = 2\varepsilon' \theta - k$ is solved as follows:

$$\begin{aligned} R' &= \left(2\varepsilon' - \frac{N}{\vec{e}^T J^{-1} \vec{e}} \right) \theta + m \\ &= m + \sigma \sqrt{\frac{2\varepsilon' \vec{e}^T J^{-1} \vec{e}}{N} - 1}, \quad (\text{B12}) \end{aligned}$$

and using Eq. (A11), we can calculate $\frac{1}{N} \vec{e}^T J^{-1} \vec{e} = \frac{1}{s^2(\alpha - 1)}$,

$$R' = m + \sigma \sqrt{\frac{2\varepsilon'}{s^2(\alpha - 1)} - 1}. \quad (\text{B13})$$

This finding is consistent with the results derived by the replica analysis in Eq. (32).

Appendix C: Replica approach for the primal problem

In this appendix, we demonstrate a replica analysis of a quenched disordered system of the primal problem. Following previous studies, when $n \in \mathbf{Z}$, the configuration average of the n th power of the partition function

$Z(R, X, \vec{r})$, $E_{X, \vec{r}}[Z^n(R, X, \vec{r})]$ is expanded as follows:

$$\begin{aligned}
& E_{X, \vec{r}}[Z^n(R, X, \vec{r})] \\
&= \frac{1}{(2\pi)^{\frac{Nn}{2} + pn}} \text{Extr}_{\vec{k}, \vec{\theta}} \int_{-\infty}^{\infty} \prod_{a=1}^n d\vec{w}_a d\vec{u}_a d\vec{v}_a \\
& E_{X, \vec{r}} \left[\exp \left(-\frac{\beta}{2} \sum_{a=1}^n \sum_{\mu=1}^p v_{\mu a}^2 + i \sum_{a=1}^n \sum_{\mu=1}^p u_{\mu a} v_{\mu a} \right. \right. \\
& \quad \left. \left. - \frac{i}{\sqrt{N}} \sum_{i=1}^N \sum_{\mu=1}^p x_{i\mu} \sum_{a=1}^n u_{\mu a} w_{ia} + \sum_{a=1}^n k_a \left(\sum_{i=1}^N w_{ia} - N \right) \right. \right. \\
& \quad \left. \left. + \sum_{a=1}^n \theta_a \left(\sum_{i=1}^N r_i w_{ia} - NR \right) \right) \right] \\
&= \frac{1}{(2\pi)^{\frac{Nn}{2} + pn}} \text{Extr}_{\vec{k}, \vec{\theta}, Q_w, \tilde{Q}_w} \int_{-\infty}^{\infty} \prod_{a=1}^n d\vec{w}_a d\vec{u}_a d\vec{v}_a \\
& \exp \left(-\frac{\beta}{2} \sum_{\mu=1}^p \sum_{a=1}^n v_{\mu a}^2 + i \sum_{\mu=1}^p \sum_{a=1}^n u_{\mu a} v_{\mu a} \right. \\
& \quad \left. - \frac{s^2}{2} \sum_{\mu=1}^p \sum_{a=1}^n \sum_{b=1}^n q_{wab} u_{\mu a} u_{\mu b} - N \sum_{a=1}^n k_a - NR \sum_{a=1}^n \theta_a \right. \\
& \quad \left. + Nm \sum_{a=1}^n \theta_a + \frac{N\sigma^2}{2} \sum_{a=1}^n \sum_{b=1}^n \theta_a \theta_b q_{wab} + \sum_{i=1}^N \sum_{a=1}^n k_a w_{ia} \right. \\
& \quad \left. - \frac{1}{2} \sum_{a=1}^n \sum_{b=1}^n \tilde{q}_{wab} \left(\sum_{i=1}^N w_{ia} w_{ib} - q_{wab} \right) \right), \quad (C1)
\end{aligned}$$

where $\vec{w}_a = (w_{1a}, \dots, w_{Na})^T \in \mathbf{R}^N$, $\vec{u}_a = (u_{1a}, \dots, u_{pa})^T \in \mathbf{R}^p$, $\vec{v}_a = (v_{1a}, \dots, v_{pa})^T \in \mathbf{R}^p$, ($a = 1, \dots, n$), and

$$E_X \left[e^{-\frac{i x_{i\mu} A}{\sqrt{N}}} \right] \simeq e^{-\frac{s^2}{2N} A^2}, \quad (C2)$$

$$E_{\vec{r}} [e^{r_i B}] \simeq e^{mB + \frac{\sigma^2}{2} B^2}, \quad (C3)$$

are used as the configuration average on $x_{i\mu}, r_i$. Moreover, $\vec{k} = (k_1, \dots, k_n)^T \in \mathbf{R}^n$, $\vec{\theta} = (\theta_1, \dots, \theta_n)^T \in \mathbf{R}^n$, $Q_w = \{q_{wab}\} \in \mathbf{R}^{n \times n}$, $\tilde{Q}_w = \{\tilde{q}_{wab}\} \in \mathbf{R}^{n \times n}$ are used. Then, as the number of assets N approaches infinity, we have

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \log E_{X, \vec{r}}[Z^n(R, X, \vec{r})] \\
&= \text{Extr}_{\vec{k}, \vec{\theta}, Q_w, \tilde{Q}_w} \left\{ \frac{1}{2} \text{Tr} Q_w \tilde{Q}_w - \vec{k}^T \vec{e} - (R - m) \vec{\theta}^T \vec{e} \right. \\
& \quad \left. + \frac{\sigma^2}{2} \vec{\theta}^T Q_w \vec{\theta} - \frac{\alpha}{2} \log \det |I + \beta s^2 Q_w| \right. \\
& \quad \left. - \frac{1}{2} \log \det |\tilde{Q}_w| + \frac{1}{2} \vec{k}^T \tilde{Q}_w^{-1} \vec{k} \right\}, \quad (C4)
\end{aligned}$$

where $\alpha = p/N \sim O(1)$, and the identity matrix $I \in \mathbf{R}^{n \times n}$ and unit vector $\vec{e} = (1, \dots, 1)^T \in \mathbf{R}^n$ are used. Then, based on the ansatz of the replica symmetry solu-

tion, with respect to $a, b = 1, 2, \dots, n$,

$$q_{wab} = \begin{cases} \chi_w + q_w & a = b \\ q_w & a \neq b \end{cases}, \quad (C5)$$

$$\tilde{q}_{wab} = \begin{cases} \tilde{\chi}_w - \tilde{q}_w & a = b \\ -\tilde{q}_w & a \neq b \end{cases}, \quad (C6)$$

$$k_a = k, \quad (C7)$$

$$\theta_a = \theta, \quad (C8)$$

are set, and

$$\begin{aligned}
\phi(R) &= \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \left\{ \lim_{N \rightarrow \infty} \frac{1}{N} \log E_{X, \vec{r}}[Z^n(R, X, \vec{r})] \right\} \\
&= \text{Extr}_{\Theta} \left\{ \frac{1}{2} (\chi_w + q_w) (\tilde{\chi}_w - \tilde{q}_w) + \frac{q_w \tilde{q}_w}{2} \right. \\
& \quad \left. - k - (R - m) \theta + \frac{\sigma^2 \theta^2}{2} \chi_w - \frac{\alpha}{2} \log(1 + \beta s^2 \chi_w) \right. \\
& \quad \left. - \frac{\alpha \beta s^2 q_w}{2(1 + \beta s^2 \chi_w)} - \frac{1}{2} \log \tilde{\chi}_w + \frac{\tilde{q}_w + k^2}{2 \tilde{\chi}_w} \right\}, \quad (C9)
\end{aligned}$$

is evaluated. Then, as the extremal conditions, we obtain

$$\frac{\partial \phi(R)}{\partial k} = -1 + \frac{k}{\tilde{\chi}_w} = 0, \quad (C10)$$

$$\frac{\partial \phi(R)}{\partial \theta} = -(R - m) + \sigma^2 \chi_w \theta = 0, \quad (C11)$$

$$\begin{aligned}
\frac{\partial \phi(R)}{\partial \chi_w} &= \frac{1}{2} (\tilde{\chi}_w - \tilde{q}_w) + \frac{\sigma^2 \theta^2}{2} - \frac{\alpha \beta s^2}{2(1 + \beta s^2 \chi_w)} \\
& \quad + \frac{\alpha \beta s^4 q_w}{2(1 + \beta s^2 \chi_w)^2} = 0, \quad (C12)
\end{aligned}$$

$$\frac{\partial \phi(R)}{\partial q_w} = \frac{1}{2} (\tilde{\chi}_w - \tilde{q}_w) + \frac{\tilde{q}_w}{2} - \frac{\alpha \beta s^2}{2(1 + \beta s^2 \chi_w)} = 0, \quad (C13)$$

$$\frac{\partial \phi(R)}{\partial \tilde{\chi}_w} = \frac{1}{2} (\chi_w + q_w) - \frac{1}{2 \tilde{\chi}_w} - \frac{\tilde{q}_w + k^2}{2 \tilde{\chi}_w^2} = 0, \quad (C14)$$

$$\frac{\partial \phi(R)}{\partial \tilde{q}_w} = -\frac{1}{2} (\chi_w + q_w) + \frac{q_w}{2} + \frac{1}{2 \tilde{\chi}_w} = 0, \quad (C15)$$

and

$$\chi_w = \frac{1}{\beta s^2 (\alpha - 1)}, \quad (C16)$$

$$q_w = \frac{\alpha}{\alpha - 1} \left(1 + \frac{(R - m)^2}{\sigma^2} \right). \quad (C17)$$

Furthermore, the minimal investment risk per asset ε is

$$\begin{aligned}
\varepsilon &= - \lim_{\beta \rightarrow \infty} \frac{\partial \phi(R)}{\partial \beta} \\
&= \lim_{\beta \rightarrow \infty} \left\{ \frac{\alpha s^2 \chi_w}{2(1 + \beta s^2 \chi_w)} + \frac{\alpha s^2 q_w}{2(1 + \beta s^2 \chi_w)^2} \right\} \\
&= \frac{s^2 (\alpha - 1)}{2} \left(1 + \frac{(R - m)^2}{\sigma^2} \right). \quad (C18)
\end{aligned}$$

Appendix D: Replica approach for the dual problem

In this appendix, we explain in detail the replica analysis of a quenched disordered system involving the dual problem. Following the discussion in the above appendix, when $n \in \mathbf{Z}$, the configuration average of n th power of the partition function $Z(\varepsilon', X, \vec{r})$, $E_{X, \vec{r}}[Z^n(\varepsilon', X, \vec{r})]$ is expanded as follows:

$$\begin{aligned} & E_{X, \vec{r}}[Z^n(\varepsilon', X, \vec{r})] \\ &= \frac{1}{(2\pi)^{\frac{Np}{2} + pn}} \text{Extr}_{\vec{k}, \vec{\theta}} \int_{-\infty}^{\infty} \prod_{a=1}^n d\vec{w}_a d\vec{u}_a d\vec{v}_a \\ & E_{X, \vec{r}} \left[\exp \left(\beta \sum_{a=1}^n \sum_{i=1}^N r_i w_{ia} + i \sum_{a=1}^n \sum_{\mu=1}^p u_{\mu a} v_{\mu a} \right. \right. \\ & \quad \left. \left. - \frac{i}{\sqrt{N}} \sum_{i=1}^N \sum_{\mu=1}^p x_{i\mu} \sum_{a=1}^n u_{\mu a} w_{ia} + \sum_{a=1}^n k_a \left(\sum_{i=1}^N w_{ia} - N \right) \right. \right. \\ & \quad \left. \left. + \sum_{a=1}^n \theta_a \left(N\varepsilon' - \frac{1}{2} \sum_{\mu=1}^p v_{\mu a}^2 \right) \right) \right]. \end{aligned} \quad (\text{D1})$$

As in the previous discussion, as the number of assets N approaches infinity, we obtain

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \log E_{X, \vec{r}}[Z^n(\varepsilon', X, \vec{r})] \\ &= \text{Extr}_{\vec{k}, \vec{\theta}, Q_w, \tilde{Q}_w} \left\{ \frac{1}{2} \text{Tr} Q_w \tilde{Q}_w - \vec{k}^T \vec{e} + \varepsilon' \vec{\theta}^T \vec{e} + n\beta m \right. \\ & \quad \left. + \frac{\sigma^2 \beta^2}{2} \vec{e}^T Q_w \vec{e} - \frac{\alpha}{2} \log \det |I + s^2 \Theta Q_w| \right. \\ & \quad \left. - \frac{1}{2} \log \det |\tilde{Q}_w| + \frac{1}{2} \vec{k}^T \tilde{Q}_w^{-1} \vec{k} \right\}, \end{aligned} \quad (\text{D2})$$

where $\Theta = \text{diag}(\theta_1, \dots, \theta_n) \in \mathbf{R}^{n \times n}$ is used. Then, based on the assumption of a replica symmetry solution, we obtain

$$\begin{aligned} \phi(\varepsilon') &= \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \left\{ \lim_{N \rightarrow \infty} \frac{1}{N} \log E_{X, \vec{r}}[Z^n(\varepsilon', X, \vec{r})] \right\} \\ &= \text{Extr}_{\Theta} \left\{ \frac{1}{2} (\chi_w + q_w) (\tilde{\chi}_w - \tilde{q}_w) + \frac{q_w \tilde{q}_w}{2} \right. \\ & \quad \left. - k + \theta \varepsilon' + \beta m + \frac{\sigma^2 \beta^2}{2} \chi_w - \frac{\alpha}{2} \log(1 + \theta s^2 \chi_w) \right. \\ & \quad \left. - \frac{\alpha \theta s^2 q_w}{2(1 + \theta s^2 \chi_w)} - \frac{1}{2} \log \tilde{\chi}_w + \frac{\tilde{q}_w + k^2}{2\tilde{\chi}_w} \right\}. \end{aligned} \quad (\text{D3})$$

Then, as the extremal conditions, we obtain

$$\frac{\partial \phi(\varepsilon')}{\partial k} = -1 + \frac{k}{\tilde{\chi}_w} = 0, \quad (\text{D4})$$

$$\begin{aligned} \frac{\partial \phi(\varepsilon')}{\partial \theta} &= \varepsilon' - \frac{\alpha s^2 \chi_w}{2(1 + \theta s^2 \chi_w)} - \frac{\alpha s^2 q_w}{2(1 + \theta s^2 \chi_w)} \\ & \quad + \frac{\alpha \theta s^4 q_w \chi_w}{2(1 + \theta s^2 \chi_w)^2} = 0, \end{aligned} \quad (\text{D5})$$

$$\begin{aligned} \frac{\partial \phi(\varepsilon')}{\partial \chi_w} &= \frac{1}{2} (\tilde{\chi}_w - \tilde{q}_w) + \frac{\sigma^2 \beta^2}{2} - \frac{\alpha \theta s^2}{2(1 + \theta s^2 \chi_w)} \\ & \quad + \frac{\alpha \theta^2 s^4 q_w}{2(1 + \theta s^2 \chi_w)^2} = 0, \end{aligned} \quad (\text{D6})$$

$$\frac{\partial \phi(\varepsilon')}{\partial q_w} = \frac{1}{2} (\tilde{\chi}_w - \tilde{q}_w) + \frac{\tilde{q}_w}{2} - \frac{\alpha \theta s^2}{2(1 + \theta s^2 \chi_w)} = 0, \quad (\text{D7})$$

$$\frac{\partial \phi(\varepsilon')}{\partial \tilde{\chi}_w} = \frac{1}{2} (\chi_w + q_w) - \frac{1}{2\tilde{\chi}_w} - \frac{\tilde{q}_w + k^2}{2\tilde{\chi}_w^2} = 0, \quad (\text{D8})$$

$$\frac{\partial \phi(\varepsilon')}{\partial \tilde{q}_w} = -\frac{1}{2} (\chi_w + q_w) + \frac{q_w}{2} + \frac{1}{2\tilde{\chi}_w} = 0, \quad (\text{D9})$$

and

$$\chi_w = \frac{1}{\theta s^2 (\alpha - 1)}, \quad (\text{D10})$$

$$q_w = \frac{\alpha}{\alpha - 1} (1 + \beta^2 \sigma^2 \chi_w^2). \quad (\text{D11})$$

Substituting the above into $\varepsilon' = \frac{\alpha s^2 \chi_w}{2(1 + \theta s^2 \chi_w)} + \frac{\alpha s^2 q_w}{2(1 + \theta s^2 \chi_w)^2}$, we then obtain

$$\varepsilon' = \frac{1}{2\theta} + \frac{s^2 (\alpha - 1)}{2} (1 + \beta^2 \sigma^2 \chi_w^2), \quad (\text{D12})$$

and

$$\beta \sigma \chi_w = \sqrt{\frac{2}{s^2 (\alpha - 1)} \left(\varepsilon' - \frac{1}{2\theta} \right)} - 1, \quad (\text{D13})$$

are evaluated. Thus, the maximal expected return per asset R' is calculated as follows:

$$\begin{aligned} R' &= \lim_{\beta \rightarrow \infty} \frac{\partial \phi(\varepsilon')}{\partial \beta} \\ &= m + \sigma \lim_{\beta \rightarrow \infty} \beta \sigma \chi_w \\ &= m + \sigma \sqrt{\frac{2\varepsilon'}{s^2 (\alpha - 1)}} - 1, \end{aligned} \quad (\text{D14})$$

where, from $\beta \sigma \chi_w \sim O(1)$, $\beta/\theta \sim O(1)$ is used. In addition, from Eq. (D12),

$$\beta \sigma \chi_w = -\sqrt{\frac{2}{s^2 (\alpha - 1)} \left(\varepsilon' - \frac{1}{2\theta} \right)} - 1, \quad (\text{D15})$$

is also derived, as $\beta \rightarrow -\infty$, the minimal expected return per asset under the investment risk is fixed, and R'' is obtained as follows:

$$R'' = m - \sigma \sqrt{\frac{2\varepsilon'}{s^2 (\alpha - 1)}} - 1. \quad (\text{D16})$$

Appendix E: Annealed disordered approach for the primal problem

Here, the typical behavior of an annealed disordered system involving the primal problem is discussed. First, following the analytical procedure of operations research, the expected investment risk $E_X [\mathcal{H}(\vec{w}|X)]$ is calculated as follows:

$$\begin{aligned} E_X [\mathcal{H}(\vec{w}|X)] &= \frac{1}{2} \vec{w}^T E_X [X X^T] \vec{w} \\ &= \frac{s^2 \alpha}{2} \sum_{i=1}^N w_i^2. \end{aligned} \quad (\text{E1})$$

Then, the object function of Lagrange's method of undetermined multipliers is prepared as follows:

$$\begin{aligned} L^{\text{OR}} &= \frac{s^2 \alpha}{2} \sum_{i=1}^N w_i^2 + k^{\text{OR}} (N - \vec{w}^T \vec{e}) \\ &\quad + \theta^{\text{OR}} (NR - \vec{w}^T \vec{r}), \end{aligned} \quad (\text{E2})$$

where the auxiliary variables $k^{\text{OR}}, \theta^{\text{OR}}$ are used. Thus, from $\frac{\partial L^{\text{OR}}}{\partial w_i} = 0$, we obtain

$$w_i = \frac{k^{\text{OR}} + \theta^{\text{OR}} r_i}{s^2 \alpha}. \quad (\text{E3})$$

Furthermore, from $\frac{\partial L^{\text{OR}}}{\partial k^{\text{OR}}} = \frac{\partial L^{\text{OR}}}{\partial \theta^{\text{OR}}} = 0$,

$$\begin{aligned} \begin{pmatrix} k^{\text{OR}} \\ \theta^{\text{OR}} \end{pmatrix} &= s^2 \alpha \begin{pmatrix} \frac{1}{N} \sum_{i=1}^N r_i & \frac{1}{N} \sum_{i=1}^N r_i^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ R \end{pmatrix} \\ &= \frac{s^2 \alpha}{m^2 + \sigma^2 - m^2} \begin{pmatrix} m^2 + \sigma^2 & -m \\ -m & 1 \end{pmatrix} \begin{pmatrix} 1 \\ R \end{pmatrix}, \end{aligned} \quad (\text{E4})$$

is assessed, where when N is sufficiently large, we have

$$\frac{1}{N} \sum_{i=1}^N r_i = m, \quad (\text{E5})$$

$$\frac{1}{N} \sum_{i=1}^N r_i^2 = m^2 + \sigma^2. \quad (\text{E6})$$

Then, the typical behavior of an annealed disordered system is obtained as follows:

$$\begin{aligned} \varepsilon^{\text{OR}} &= \frac{s^2 \alpha}{2} \frac{1}{N} \sum_{i=1}^N \left(\frac{k^{\text{OR}} + \theta^{\text{OR}} r_i}{s^2 \alpha} \right)^2 \\ &= \frac{1}{2 s^2 \alpha} ((k^{\text{OR}})^2 + 2 k^{\text{OR}} \theta^{\text{OR}} m + (\theta^{\text{OR}})^2 (m^2 + \sigma^2)) \\ &= \frac{s^2 \alpha}{2} \left(1 + \frac{(R - m)^2}{\sigma^2} \right). \end{aligned} \quad (\text{E7})$$

Appendix F: Annealed disordered approach for the dual problem

Here, the typical behavior of an annealed disordered system involving the dual problem is also discussed. First, followed by the analytical procedure of operations research, the object function of Lagrange's method of undetermined multipliers is defined as follows:

$$\begin{aligned} L^{\text{OR}} &= \vec{r}^T \vec{w} + k^{\text{OR}} (\vec{w}^T \vec{e} - N) \\ &\quad + \theta^{\text{OR}} \left(N \varepsilon' - \frac{s^2 \alpha}{2} \sum_{i=1}^N w_i^2 \right), \end{aligned} \quad (\text{F1})$$

where, as in the primal problem discussed above, the expected investment risk in Eq. (E1), $\frac{s^2 \alpha}{2} \sum_{i=1}^N w_i^2$ is used. Thus, from $\frac{\partial L^{\text{OR}}}{\partial w_i} = 0$, we obtain

$$w_i = \frac{k^{\text{OR}} + r_i}{s^2 \alpha \theta^{\text{OR}}}. \quad (\text{F2})$$

Moreover, from $\frac{\partial L^{\text{OR}}}{\partial k^{\text{OR}}} = \frac{\partial L^{\text{OR}}}{\partial \theta^{\text{OR}}} = 0$, we obtain

$$1 = \frac{k^{\text{OR}} + m}{s^2 \alpha \theta^{\text{OR}}}, \quad (\text{F3})$$

$$\begin{aligned} \varepsilon' &= \frac{s^2 \alpha}{2 s^4 \alpha^2 (\theta^{\text{OR}})^2} ((k^{\text{OR}})^2 + 2 k^{\text{OR}} m + m^2 + \sigma^2) \\ &= \frac{s^2 \alpha}{2} \left(1 + \left(\frac{\sigma}{s^2 \alpha \theta^{\text{OR}}} \right)^2 \right). \end{aligned} \quad (\text{F4})$$

Based on this and $R' = \frac{1}{N} \sum_{i=1}^N r_i \left(\frac{k^{\text{OR}} + r_i}{s^2 \alpha \theta^{\text{OR}}} \right) = \frac{k^{\text{OR}} m + m^2 + \sigma^2}{s^2 \alpha \theta^{\text{OR}}}$, we obtain

$$\begin{aligned} R' &= m + \sigma \frac{\sigma}{s^2 \alpha \theta^{\text{OR}}} \\ &= m + \sigma \sqrt{\frac{2 \varepsilon'}{s^2 \alpha}} - 1, \end{aligned} \quad (\text{F5})$$

where, from Eq. (F4), we use

$$\frac{\sigma}{s^2 \alpha \theta^{\text{OR}}} = \sqrt{\frac{2 \varepsilon'}{s^2 \alpha}} - 1. \quad (\text{F6})$$

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